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Nonlinear effects of gravitational and electromagnetic radiation on the propagation of light

B Linet

Equipe de Recherche Associée au CNRS No 533, Université Paris VI, Institut Henri Poincaré, 11, rue P et M Curie, 75231 Paris Cedex 05, France

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Abstract. We determine the frequency shift between the emitter and the receiver due to nonlinear effects of gravitational and electromagnetic radiation from a bounded source.

1. Introduction

In astrophysics it is known that intensity and frequency fluctuations of the observed light, emitted by a distant object, may arise from gravitational radiation (Zipoy 1966, Zipoy and Bertotti 1968, Bergmann 1971, Burke 1975, Sazhin 1978). In the case of a single bounded source, these effects on the light propagation are investigated in the far zone of the gravitational field of this source within the framework of the first approximation of general relativity. Yet, since the bounded source is assumed to be quasi-periodic, the linear effects are null when they are averaged over the time. Therefore, one often considers random gravitational waves. In this paper, we shall study the nonlinear effects of the gravitational radiation of a bounded source on the light propagation in the case of geometrical optics.

The method of this approach also enables us to show the influence of electromagnetic radiation from a bounded source on the propagation of electromagnetic waves. In the Einstein–Maxwell theory, the asymptotic gravitational field depends on the electromagnetic radiation of the bounded source. Then, we have an indirect effect of the electromagnetic field on the light propagation. Although we consider the light as a test electromagnetic field, we obtain, in a certain sense, an example of nonlinearity in electromagnetism resulting from the Einstein–Maxwell theory.

In order to be in a position to calculate these effects, we have to know the asymptotic gravitational field of a bounded source in the Einstein–Maxwell theory. Also we shall recall in § 2 some results which will be needed. Then we shall determine, in § 3, the equation giving the frequency shift of the light by nonlinear effects when the light rays move in the asymptotic gravitational field. In § 4, we shall perform the integration of this equation for a simple model of gravitational and electromagnetic radiation. We shall specify how the frequency shift is related to the impact parameter. This result might be applicable to the case of two images of the same object formed by a gravitational lens (Walsh *et al* 1979). However the effect is probably unobservable.

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2. Asymptotic gravitational field

The study of gravitational radiation in a space–time with a bounded source is well described in the formalisms of Bondi *et al* (1962), Sachs (1962) and Newman and Penrose (1962). They work with an asymptotic expansion in $1/r$ along the null coordinate u. Yet, for some questions, it is more practical to use a coordinate system in which the metric is expressed asymptotically as in cartesian-like coordinates. Therefore Papapetrou (1969) has introduced a coordinate system (x^0, x^i) which is related to the Newman and Penrose coordinates by the transformation

$$\begin{aligned} x^0 &= u + r, \qquad & x^1 &= r \sin\theta \sin\varphi, \\ x^2 &= r\sin\theta\cos\varphi, \qquad & x^3 &= r\cos\theta. \end{aligned} \tag{1}$$

Thus when the gravitational field is absent the trajectories of the light rays have a simple expression in this coordinate system.

The development of the metric has the following form:

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{\infty} \frac{1}{r^n} \overset{(n)}{g}_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} + \sum_{n=1}^{\infty} \frac{1}{r^n} \overset{(n)}{g}{}^{\mu\nu}, \tag{2}$$

with

$$\overset{(n)}{g}_{\mu\nu}(u, \theta, \varphi) \text{ and } \overset{(n)}{g}{}^{\mu\nu}(u, \theta, \varphi).$$

We note that

$$\overset{(1)}{g}{}^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\mu\beta}\overset{(1)}{g}_{\alpha\beta}. \tag{3}$$

Papapetrou (1969) has studied the development (2) of the metric. A tetrad formalism is introduced and only three vectors are necessary in order to express the metric. These are

$$\begin{aligned} &l^\mu: (1, x^i/r) \\ &a^\mu: (0, \cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\ &b^\mu: (0, -\sin\varphi, \cos\varphi, 0). \end{aligned} \tag{4}$$

The general form of the metric is

$$\begin{aligned} g^{\mu\nu} = \eta^{\mu\nu} &+ a'(a^\mu a^\nu - b^\mu b^\nu) + b'(a^\mu b^\nu + a^\nu b^\mu) + c'(a^\mu a^\nu + b^\mu b^\nu) \\ &+ A'(l^\mu a^\nu + l^\nu a^\mu) + B'(l^\mu b^\nu + l^\nu b^\mu) + C' l^\mu l^\nu. \end{aligned} \tag{5}$$

The development (2) of $g^{\mu\nu}$ induces a similar development of the six quantities $a', \ldots$. We can express $\overset{(1)}{a}{}', \ldots$ with the help of the Newman and Penrose quantities. We obtain

$$\begin{aligned} &\overset{(1)}{a}{}' = \sigma^0 + \bar\sigma^0, \qquad \overset{(1)}{b}{}' = -\mathrm{i}(\sigma^0 - \bar\sigma^0), \qquad \overset{(1)}{c}{}' = 0, \\ &\overset{(1)}{A}{}' = -\frac{1}{\sqrt{2}}(\omega^0 + \bar\omega^0), \qquad \overset{(1)}{B}{}' = \frac{\mathrm{i}}{\sqrt{2}}(\omega^0 - \bar\omega^0), \qquad \overset{(1)}{C}{}' = -(\psi_2^0 + \bar\psi_2^0), \end{aligned} \tag{6}$$

where ω^0 has the following expression:

$$\omega^0 = \frac{1}{\sqrt{2}}(\bar{\mathscr{D}} + 2\cot\theta)\sigma^0 \qquad \text{with } \mathscr{D} = \frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}.$$

On the other hand, the equation which governs the time-dependence of ψ_2^0 is

$$\dot{\psi}_2^0 = -\tfrac{1}{2}(\mathscr{D} + \cot\theta)(\mathscr{D} + 2\cot\theta)\dot{\bar{\sigma}}^0 - \sigma^0\ddot{\bar{\sigma}}^0 \tag{7}$$

where the dot means the derivative with respect to u. We have now completely determined the asymptotic gravitational field in terms of the news function $\sigma^0(u, \theta, \varphi)$.

In the Einstein–Maxwell theory, Kozarzewski (1965) and Exton *et al* (1969) have studied the asymptotic gravitational field in the Newman–Penrose formalism. It is a simple matter to express $\overset{(1)}{g}{}^{\mu\nu}$ in this case. We obtain the same expressions for $\overset{(1)}{a}{}', \ldots$ in terms of the Newman–Penrose quantities. However we have to replace equation (7) by the following equation for the time evolution of ψ_2^0:

$$\dot{\psi}_2^0 = -\tfrac{1}{2}(\mathscr{D} + \cot\theta)(\mathscr{D} + 2\cot\theta)\dot{\bar{\sigma}}^0 - \sigma^0\ddot{\bar{\sigma}}^0 + \phi_2^0\bar{\phi}_2^0 \tag{8}$$

where $\phi_2^0(u, \theta, \varphi)$ is the news function for the electromagnetic field.

3. Light propagation

We are interested in determining the influence of the asymptotic gravitational field, described above, on the light propagation in the limit of geometrical optics using a method of approximations.

As a zero-order approximation (when the gravitational field is absent), we assume that the trajectory of the light ray is located in the plane $0x^1x^2$ with an impact parameter $x^2 = D$. The parametrised form of the path of this non-perturbed light ray is

$$\overset{(0)}{x}{}^0 = \frac{1}{c}\omega\lambda, \qquad \overset{(0)}{x}{}^1 = \frac{1}{c}\omega\lambda, \qquad \overset{(0)}{x}{}^2 = D, \qquad \overset{(0)}{x}{}^3 = 0, \tag{9}$$

where we have introduced the frequency ω of the electromagnetic wave. The tangent vector field of the curve (9) is

$$\overset{(0)}{k}{}^\beta = (1/c)\omega(1, 1, 0, 0). \tag{10}$$

In curved space–time the rays are geodesics. It is convenient to use the following form of geodesic equation for the tangent vector field of the geodesic line:

$$k^\beta\partial_\beta k_\alpha - \tfrac{1}{2}\partial_\alpha g_{\mu\nu}k^\mu k^\nu = 0. \tag{11}$$

In the asymptotic gravitational field of the bounded source, the light ray is clearly a perturbation of the straight line (9). Therefore we let

$$k_\alpha = \overset{(0)}{k}_\alpha + \delta k_\alpha. \tag{12}$$

When we introduce the development (2) of the metric in equation (11), we assume that the first term of order $1/r$ yields the equation for δk_α. Also we have

$$\overset{(0)}{k}{}^\beta\partial_\beta\delta k_\alpha - \tfrac{1}{2}\partial_\alpha[(1/r)\overset{(1)}{g}_{\mu\nu}]\overset{(0)}{k}{}^\mu\overset{(0)}{k}{}^\nu = 0. \tag{13}$$

The aim of this paper is to investigate from equation (13) the change in frequency of the light between the emitter and the receiver. We suppose that they are both at rest with respect to the bounded source. Moreover, at their positions, neglecting the gravitational field, the frequency is given by k_0. Taking into account (3), we can write the equation (13) for $\alpha = 0$ in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta k_0 + \frac{1}{2r}\partial_0\left(\overset{(1)}{g}{}^{\mu\nu}\overset{(0)}{k}_\mu \overset{(0)}{k}_\nu\right) = 0. \tag{14}$$

The expression in parentheses in (14) is calculated with the help of formula (5); we obtain

$$\overset{(1)}{g}{}^{\mu\nu}\overset{(0)}{k}_\mu \overset{(0)}{k}_\nu = \left(\frac{\omega}{c}\right)^2\left[(\cos\varphi - 1)^2 \overset{(1)}{C}{}' - 2\sin\varphi(\cos\varphi - 1)\overset{(1)}{B}{}' - \sin^2\varphi\, \overset{(1)}{a}{}'\right] \tag{15}$$

where $\overset{(1)}{C}{}'$, $\overset{(1)}{B}{}'$ and $\overset{(1)}{a}{}'$ are given in (6).

Since we suppose that the news function σ^0 is quasi-periodic, only nonlinear effects in σ^0 could give a contribution to δk_0 which is not null when averaged over the time. Eliminating linear terms in σ^0 and using the time-evolution equation (8), we deduce from (14) the following equation for δk_0:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta k_0 + \frac{1}{2}\left(\frac{\omega}{c}\right)^2 \frac{(\cos\varphi - 1)^2}{r}(\sigma^0\ddot{\bar{\sigma}}{}^0 + \bar{\sigma}^0\ddot{\sigma}^0 - 2\phi_2^0\bar{\phi}_2^0), \tag{16}$$

in which all quantities are calculated on the straight line (9) and therefore depend only on λ.

4. Frequency shift

In order to integrate equation (16), we must specify the news functions σ^0 and ϕ_2^0. As we seek only an order of magnitude of the change in frequency, we shall calculate it for a simple model of quadrupole gravitational radiation and dipole electromagnetic radiation that we shall suppose time periodic and axisymmetric with respect to the axis $0x^3$.

We recall that if P_{G} and P_{E} are the radiative gravitational and electromagnetic powers, respectively, we have the general formula

$$\frac{G}{c^5}(P_{\mathrm{G}} + P_{\mathrm{E}}) = \frac{1}{4\pi}\int_{(S)} (\dot{\sigma}^0\dot{\bar{\sigma}}{}^0 + \phi_2^0\bar{\phi}_2^0)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{17}$$

where G is Newton's gravitational constant. Consequently, for our simple model of radiation we can express σ^0 and ϕ_2^0 in terms of P_{G} and P_{E}. We find

$$\begin{aligned}\dot{\sigma}^0 &= \frac{1}{2}\left(\frac{15}{2}\right)^{1/2}\left(\frac{GP_{\mathrm{G}}}{c^5}\right)^{1/2}\sin^2\theta\exp(\mathrm{i}\omega_{\mathrm{g}}u),\\ \phi_2^0 &= \left(\frac{3}{2}\right)^{1/2}\left(\frac{GP_{\mathrm{E}}}{c^5}\right)^{1/2}\sin\theta\exp(\mathrm{i}\omega_{\mathrm{e}}u).\end{aligned} \tag{18}$$

With the news functions (18), equation (16) can be integrated in the form

$$\delta k_0(\lambda_2) - \delta k_0(\lambda_1) = \frac{G}{c^5}\left(\frac{15}{8}P_{\mathrm{G}} + \frac{3}{2}P_{\mathrm{E}}\right)\frac{1}{D}\left(\frac{\omega}{c}\right)^2\int_{\lambda_1}^{\lambda_2}(\cos\varphi - 1)^2\sin\varphi\,\mathrm{d}\lambda \tag{19}$$

where λ_1 and λ_2 are respectively the values of the parameter λ at the emitter and at the receiver. We denote by $-L_1$ and L_2 respectively the position of the axis $0x^1$ of the emitter and of the receiver. By integrating we obtain the frequency shift in closed form:

$$z = \delta\omega/\omega = (G/c^5)(\tfrac{15}{8}P_G + \tfrac{3}{2}P_E)I(L_1, L_2; D) \tag{20}$$

with

$$I(L_1, L_2; D) = 2\ln\left(1 + \frac{L_2}{(L_2^2 + D^2)^{1/2}}\right) - 2\ln\left(1 - \frac{L_1}{(L_1^2 + D^2)^{1/2}}\right) - \frac{L_2}{(L_2^2 + D^2)^{1/2}} - \frac{L_1}{(L_1^2 + D^2)^{1/2}}. \tag{21}$$

One can verify that $I(L_1, L_2; D)$ is a positive function. Therefore we see from (20) that the frequency shift is always a blueshift. In a way the photon gains energy by nonlinear interaction with gravitational and electromagnetic radiation of a bounded source.

We remark that if $L_1 \to \infty$ the expression (21) is divergent. This fact is not surprising because we have considered a time-periodic bounded source. Indeed, during an infinite interval of time, it radiates an infinite amount of energy. Physically we must not take $L_1 \to \infty$ for this case. On the other hand, when $L_2 \to \infty$ the situation is different because the light tends to propagate in the direction of the gravitational wave.

Now, we consider two light rays in the plane $0x^1x^2$ having two different impact parameters D and D' for the same distances L_1, and L_2. We can express the corresponding frequency shifts z and z' with the following assumptions:

$$D/L_1 \ll 1, \qquad D/L_2 \ll 1, \qquad D'/L_1 \ll 1, \qquad D'/L_2 \ll 1. \tag{22}$$

An asymptotic expression of the expression (21) yields the formula

$$z - z' = -(4G/c^5)(\tfrac{15}{8}P_G + \tfrac{3}{2}P_E)\ln(D/D'). \tag{23}$$

From (23), we see that the light ray which is situated closer to the bounded source appears to be redshifted with respect to the other one.

5. Conclusion

The frequency shift, given by the formulae (20) and (23), has been determined for a simple model of gravitational and electromagnetic radiation of the bounded source. In a more general case, it is reasonable to expect that the effects will be qualitatively the same.

The effect (23) might be interesting in the case of a single object which has been split into two images by a gravitational lens (Walsh *et al* 1979). However, this effect is exceedingly small. To obtain a relative frequency shift of 10^{-5}, the bounded source must radiate 10^7 solar masses per year. Of course, the essential part of the relative frequency shift is the electromagnetic contribution. Thus, we can neglect the linear and nonlinear effects in σ^0 when we calculate the frequency shift. Consequently, we remark that the relative time delay due to the difference in path length does not influence it.

An appreciably more favourable situation has to be found, perhaps by considering a more realistic radiative bounded source.

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